On the Order of Deferred Correction

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Abstract

New deferred correction methods for the numerical solution of initial value problems in ordinary differential equations have recently been introduced by Dutt, Greengard and Rokhlin. A convergence proof is presented for these methods, based on the abstract Stetter-Lindberg-Skeel framework and Spijker-type norms. It is shown that p corrections of an order-r one-step solver yield order $r(p+1)$ accuracy.

1 Introduction.

Deferred correction methods for the numerical solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(a) = y_a \in \mathbb{R}^d, \quad t \in [a, b] \tag{1.1}$$

have been developed and analyzed for many years [3]. There are two important features of such methods: firstly, the ability to easily estimate the global error and secondly the ability to easily create high order methods from low order schemes. The combination of these powerful qualities opens up the possibility for sophisticated codes with adaptive step-sizes and also adaptive order. Two interesting new techniques of deferred correction were introduced in [6]. The first method (with the slightly misleading name) "classical" Deferred Correction is a method that is similar to Zadunaisky's Iterated Defect Correction (IDeC) [16]. It is based on a previously numerically computed solution to (1.1) where, after deriving and solving an ODE (numerically) for the global error of the computed solution, the numerical estimate of the error is added to the previously computed solution. The second method named Spectral Deferred Correction is based on the same ideas, however, the ODE for the error is turned into an integral equation. The extensive previous convergence theory of deferred correction methods does not apply to these new techniques (and thus there are no formal proofs justifying the convergence of these new methods), however, in this paper, we extend and apply previous technical tools to prove high-order convergence for the first ("classical") deferred correction scheme of [6]. Convergence of Spectral Deferred Correction is proved in [10], and the method has been successfully implemented and tested in [6, 13, 11].

Previous convergence proofs [12, 7, 8] for deferred correction methods often assume a global asymptotic error expansion, which Runge-Kutta methods usually possess (but multistep methods usually lack). Our proof relies instead on the smoothness of the global error in discrete Sobolev norms, defined via divided differences as in [14], and adjusted to fit our situation. Our approach is modeled on the abstract Stetter-Lindberg-Skeel error analysis [15, 12, 14], which treats the initial value problem (IVP) as an operator equation approximated by a discrete operator equation. Lindberg and Skeel used this approach to

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develop new methods and show convergence of new and existing methods. Skeel extended it into a very general framework for the analysis of accuracy and convergence, which permits the analysis of many deferred correction methods. We extend it further to prove convergence for the new classical scheme of [6]. One should note that our convergence analysis only covers one-step methods, however, the numerical examples in Section 5 suggest that multistep methods will also work, although with some restrictions. We hope that our "global asymptotic error" free framework can (in the future) be adapted to include multistep methods.

2 The DGR scheme

The first method introduced in [6] is referred to as "classical deferred correction" . We believe that this name is a little misleading as the method is not classical at all. It is quite similar to Zadunaisky's iterated deferred correction (IDeC) [16], however, a different method. We will therefore refer to this method as the DGR method (or scheme) after its inventors Dutt, Greengard and Rokhlin.

The idea of the method is as follows. One constructs a new IVP for the error, solves it numerically, and thus obtains an approximation to the global error which is added to the previous numerical solution. The process is repeated on each subinterval separately, and can be viewed as a technique for generating high-order Runge-Kutta-Fehlberg schemes, without the laborious algebra required to solve large nonlinear systems of order conditions.

2.1 Description of the algorithm

Suppose a numerical solution $u = (u_0, \ldots, u_n)$ is given at (equidistant) grid points $\{t_0, \ldots, t_n\}$ with step size h on the current subinterval $[a, b]$ of time integration, with error

$$y(t_k) - u_k = \mathcal{O}(h^p), \qquad k = 0, \ldots, n.$$

We can view u as a continuous approximate solution satisfying the IVP with error

$$\tilde{\delta}(t) = y(t) - \nabla_n u(t), \qquad t \in [a, b] \tag{2.1}$$

by employing the Lagrange interpolation operator $\nabla_n : \mathbb{R}^{n+1} \to C[a, b]$ based on the grid points. Differentiating the error formula (2.1) and using the IVP for y gives the error equation

$$\tilde{\delta}'(t) = \tilde{f}(t, \tilde{\delta}(t)), \qquad \tilde{\delta}(a) = \tilde{\delta}_a = y(a) - u_0, \tag{2.2}$$

$$\tilde{f}(t, \tilde{\delta}(t)) = f\left(t, \tilde{\delta}(t) + \nabla_n u(t)\right) - \frac{d}{dt}\nabla_n u(t).$$

Solving (2.2) with an order-p accurate numerical method gives a numerical error

$$\delta_k = y(t_k) - u_k + \mathcal{O}(h^{2p}), \qquad k = 0, \ldots, n.$$

The procedure can be iterated, using the same order-p method at each iteration, or more generally we may use one of r different one-step methods of orders $p_1, \ldots, p_r$ at each correction. We refer to this approach as the DGR algorithm:

Algorithm 2.1. For j =1:r

- Interpolate $u^{[j-1]} \to \nabla_n u^{[j-1]}(t)$ on the current subinterval

- Solve the following IVP by p_j-th order method:

$$\tilde{\delta}'(t) = f(t, \tilde{\delta}(t) + \nabla_n u^{[j-1]}(t)) - \frac{d}{dt}\nabla_n u^{[j-1]}(t), \quad \tilde{\delta}(a) = y(a) - \nabla_n u^{[j-1]}(a),$$

  to get a numerical approximation δ to the current error $\tilde{\delta}$

- Update $u^{[j]} = u^{[j-1]} + \delta$

end

The similarity with the IDeC methods is obvious, however, it is clear that the DGR method is different from the IDeC scheme. This can be seen in the following way. The IDeC method is based on a computed numerical solution $u = (u_0, \ldots, u_n)$. One then forms the defect $d(t) = \frac{d}{dt}\nabla_n u(t) - f(t, \nabla_n u(t))$, solves

$$\tilde{\rho}'(t) = f(t, \tilde{\rho}(t)) + d(t), \qquad \tilde{\rho}(a) = y_a$$

numerically and obtain an approximation $\rho = \{\rho_0, \ldots, \rho_n\}$. The update is defined by

$$u_\nu^{\text{new}} = u_\nu - (\rho_\nu - \nabla_n u(t_\nu)), \qquad \nu = 0, \ldots, n.$$

We observe that in the DGR method the error equation (2.2) is approximated in a more direct way, without explicit use of the defect, which makes a real difference especially for nonlinear problems. It is therefore clear that when f is non-linear, the methods are very different. It appears that the DGR scheme allows for some extra freedom when constructing high order methods compared to the IDeC method. In particular, one can mix one-step methods of completely arbitrary orders and, as suggested in Section 5.4, one can even combine multistep methods with one-step methods. The proof of this rather striking phenomenon is not covered in this paper however, and this is left for future research.

The reader will observe, both in the proof and the numerical examples, that the requirement of an equidistant grid is crucial, and this is also the case for IDeC. Improved variants of IDeC that do not require equidistant grid have been invented and analyzed in [1, 2], and it is an open question whether one can apply these ideas to the DGR method. If so, it appears that one may have a very general way of creating high order methods from lower order one-step and multistep methods that allow for variable step sizes.

3 Theoretical framework

We employ Stetter's abstract formalism for analyzing numerical solutions of the IVP [15], as do most previous analyses of deferred correction [14, 12]. Thus we write the IVP (1.1) as an operator equation

$$Fy = 0 \tag{3.1}$$

where $F : Y \to Z$ is an operator between normed linear spaces Y and Z. A numerical method ϕ for the IVP approximates (3.1) by a family of operator equations

$$\phi_n(F)u = 0, \qquad n \in \mathbb{N},$$

where $\phi_n(F) : Y_n \to Z_n$ is an operator between finite-dimensional normed linear spaces Z_n and Y_n with dimensions proportional to n. Euler's method, for example (applied to a one-dimensional IVP), has $Y_n = Z_n = \mathbb{R}^{n+1}$ and

$$\phi_n(F)(u)_\nu = \begin{cases} -u_0 + y_a & \nu = 0 \\ -\frac{u_\nu - u_{\nu-1}}{t_\nu - t_{\nu-1}} + f(t_{\nu-1}, u_{\nu-1}) & \nu = 1, \ldots, n, \end{cases}$$

where $\{t_0, \ldots, t_n\}$ is a grid on $[a, b]$ and y_a is the initial value (we will be more rigorous with the definitions below). Stetter requires Y and Z to be Banach spaces. It turns out that completeness is not usually necessary for analyzing the convergence of most numerical methods $\phi_n(F)$, so normed linear spaces suffice. (Since Example 1.1 of [15] is not a Banach space, it is fortunate that normed linear spaces suffice.) Following Skeel [14], we convert the IVP (1.1) to an operator equation with the norm

$$\|z\|_{Y^m} := \max\{\|z\|_\infty, \frac{1}{2!}\|z'\|_\infty, \ldots, \frac{1}{m!}\|z^{[m]}\|_\infty\} \tag{3.2}$$

on the normed linear space $Y^m := C^m[a, b] \times \cdots \times C^m[a, b]$ (d times). Here we use the standard maximum norm

$$\|z\|_\infty := \max_{t\in[a,b],1\le j\le d} |z_j(t)|, \qquad z_j \in C^m[a, b].$$

The operator F is defined so that $Fy = 0$ is equivalent to the IVP: Thus, for $z \in Y^m$

$$Fz(t) := \big(-z(a) + y_a, -z'(t) + f(t, z(t))\big), \qquad t \in [a, b], \tag{3.3}$$

where $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed smooth with bounded derivatives. Then the initial value y_a in F guarantees that y is the unique solution of the IVP. The range of F is then naturally defined to be $Z^m := \mathbb{R}^d \times C^{m-1}[a, b] \times \cdots \times C^{m-1}[a, b]$ (d times), and for $g = (g_a, \tilde{g}) \in Z^m$ we define the norm by

$$\|g\|_{Z^m} := \|Lg\|_{Y^m}, \qquad Lg(t) = g_a + \int_a^t \tilde{g}(s)\, ds. \tag{3.4}$$

3.1 The discrete problem

Finite-dimensional spaces for the numerical method are built on a grid on the interval $[a, b]$. In particular, for $n \in \mathbb{N}$, we let

$$\mathbb{G}_n = \{t_0, \ldots, t_n : a = t_0 < \cdots < t_{\nu-1} < t_\nu < t_n = b\}, \tag{3.5}$$

with step sizes $h_\nu := t_\nu - t_{\nu-1} > 0$ for $\nu = 1, \ldots, n$, and mesh size $h := \max_\nu\{h_\nu\}$. Given a grid $\mathbb{G}_n$, we define finite-dimensional spaces $Y_n^m = Z_n^m := \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ ($n + 1$ times). We will interpret $u \in Y_n^m, Z_n^m$ as a column vector in $\mathbb{R}^{d(n+1)}$, where (with a slight abuse of notation) $u_\nu \in \mathbb{R}^d$ is interpreted as the vector corresponding to t_ν. In particular,

$$u = ((u_0)_1, (u_0)_2, \ldots, (u_0)_d, (u_1)_1, (u_1)_2, \ldots, (u_1)_d, \ldots, (u_n)_1, (u_n)_2, \ldots, (u_n)_d)^T.$$

The definition of the point evaluation operators (or grid mappings, as we will also call them) should make this clear. In particular, $\Delta_n : Y^m \to Y_n^m$ and $\Lambda_n : Z^m \to Z_n^m$ are defined by

$$(\Delta_n z)_\nu := z(t_\nu), \qquad z \in Y^m, \quad 0 \le \nu \le n, \tag{3.6}$$

and

$$(\Lambda_n g)_\nu := \begin{cases} g_a & \nu = 0, \\ g(t_{\nu-1}) & 1 \le \nu \le n, \end{cases} \qquad g \in Z^m, \tag{3.7}$$

where $t_\nu, t_{\nu-1} \in \mathbb{G}_n$. The norms on Y^m and Z^m suggest natural norms on Y_n^m and Z_n^m, with backward divided differences replacing derivatives. For $u \in Y_n^m$ let

$$\|u\|_{Y_n^m} := \max\{\|u\|_\infty, \|\tilde{D}u\|_\infty, \ldots, \|\tilde{D}^m u\|_\infty\}, \tag{3.8}$$

where

$$\|u\|_\infty = \max_{0\le\nu\le n,1\le j\le d} |(u_\nu)_j|,$$

and

$$\tilde{D}^m u = (u_0, Du_1, D^2u_2, \ldots, D^m u_m, \ldots, D^m u_n)^T$$

is an array of divided differences and initial values. Here D^m denotes the backward divided difference defined by

$$D^0 u_\nu = u_\nu,$$

$$D^m u_\nu = \frac{D^{m-1}u_\nu - D^{m-1}u_{\nu-1}}{t_\nu - t_{\nu-m}}, \quad m = 1, 2, \ldots., \quad m \le \nu.$$

For $z \in Z_n^m$ we define

$$\|z\|_{Z_n^m} := \|(L_h \otimes I_d)z\|_{Y_n^m}, \tag{3.9}$$

where

$$L_h := \begin{bmatrix} 1 & & & & \\ 1 & h_1 & & & \\ 1 & h_1 & h_2 & & \\ \vdots & \vdots & \vdots & & \\ 1 & h_1 & h_2 & \ldots & h_n \end{bmatrix}, \qquad I_d x = x, \quad x \in \mathbb{R}^d.$$

Thus, L_h works as a discrete integral operator. In particular,

$$\|z\|_{Z_n^0} = \max_{0 \le l \le n, 1 \le j \le d} |(z_0 + \sum_{k=1}^{l} h_k z_k)_j|$$

is often called the Spijker norm, while

$$\|z\|_{Z_n^1} = \max_{1 \le j \le d} \{|(z_0)_j|, |(z_1)_j|, |(z_2)_j|, \ldots, |(z_n)_j|\}$$

is the usual maximum norm. These spaces and mappings are related by the following "asymptotically commutative" diagram:

$$\begin{array}{ccc} Y^m & \xrightarrow{F} & Z^m \\ {\scriptstyle \Delta_n}\downarrow & & \downarrow{\scriptstyle \Lambda_n} \\ Y_n^m & \xrightarrow{\phi_n(F)} & Z_n^m. \end{array}$$

In the sequel we will need the following presumably well-known lemma, whose proof we include for completeness.

Lemma 3.1. *Let $n, m \in \mathbb{N}$ with $n \ge m$. For an equidistant grid $\mathbb{G}_n$ on $[a, b]$, the Lagrange interpolation operator $\nabla_n : Y_n^m \to Y^m$ satisfies a norm bound $\|\nabla_n\| \le C_n$ where C_n depends only on n and $[a, b]$. Moreover, the statement is valid (with the same C_n) for any interval $[\tilde{a}, \tilde{b}] \subset [a, b]$.*

Proof. Let $\zeta \in Y_n^m$. By Newton's interpolation formula, we have

$$\nabla_n \zeta(t) = \sum_{\mu=0}^{n} D^\mu \zeta_\mu \pi_\mu(t), \qquad \pi_\mu(t) = \prod_{l=0}^{\mu-1}(t - t_l), \quad \mu \ge 1, \quad \pi_0(t) = 1,$$

which gives

$$\|\nabla_n \zeta\|_{Y^m} \le (n+1) \max_{0 \le \mu \le n} \{\|D^\mu \zeta_\mu\|_{\mathbb{R}^d} \|\pi_\mu\|_{Y^m}\},$$

where the norm $\|\cdot\|_{\mathbb{R}^d}$ is chosen to be the max norm. Since $\mathbb{G}_n$ is equidistant, it follows that

$$\|\pi_{m+k}\|_{Y^m} \leq \hat{C}_n h^k, \quad k = 0, \ldots, n-m, \qquad h = t_l - t_{l-1},$$

where $\hat{C}_n$ depends only on n and $[a, b]$, however, the bound is valid for any interval $[\tilde{a}, \tilde{b}] \subset [a, b]$. Also, $\|D^{m+k}\zeta_{m+k}\|_{\mathbb{R}^d} \leq K_n \|\zeta\|_{Y_n^m}/h^k$ for some $K_n > 0$ depending on n and independent of $[a, b]$. Hence,

$$\|D^\mu \zeta_{m+k}\|_{\mathbb{R}^d} \|\pi_{m+k}\|_{Y^m} \leq \widetilde{C}_n \|\zeta\|_{Y_n^m}, \qquad k = 1, \ldots, n-m, \tag{3.10}$$

for some $\widetilde{C}_n > 0$ depending on n and $[a, b]$, however still true for any $[\tilde{a}, \tilde{b}] \subset [a, b]$. Now, if $\widetilde{K}_n$ bounds $\|\pi_\mu\|_{Y_n^m}$, we obviously have

$$\max_{0 \leq \mu \leq m} \{\|D^\mu \zeta_\mu\|_{\mathbb{R}^d} \|\pi_\mu\|_{Y_n^m}\} \leq \widetilde{K}_n \|\zeta\|_{Y_n^m}, \tag{3.11}$$

where the latter equation would be true for any $[\tilde{a}, \tilde{b}] \subset [a, b]$. Thus, (3.10) and (3.11) yield the existence of $C_n > 0$ as asserted in the lemma such that $\|\nabla_n \zeta\|_{Y_n^m} \leq C_n \|\zeta\|_{Y_n^m}$. □

We plan to prove convergence of deferred correction schemes indirectly, by using stability and consistency in the usual manner. However, the following convenient and effective definitions from [15] may differ from the many other definitions of consistency and stability in the literature.

Definition 3.2. *The sequence*

$$\lambda_n = \phi_n(F)\Delta_n y \in Z_n^m, \quad n \in \mathbb{N}$$

where y is the solution of the IVP $Fy = 0$, where F is defined in (3.3), is called the local discretization error.

Definition 3.3. *A discretization method ϕ is called stable if there exist a positive constant S, independent of n, such that*

$$\|v - w\|_{Y_n^m} \leq S \|\phi_n(F)v - \phi_n(F)w\|_{Z_n^m} \tag{3.12}$$

for all $v, w \in Y_n^m$.

3.2 Local error for one-step methods

Runge-Kutta methods have been widely used in deferred correction algorithms [7, 8]. While we do not restrict our analysis to Runge-Kutta methods we assume that the method ϕ is of the form

$$\phi_n(F)(u)_\nu = \begin{cases} -u_0 + y_a & \nu = 0 \\ -\frac{u_\nu - u_{\nu-1}}{h} + \Psi(h, u_{\nu-1}, f) & \nu = 1, \ldots, n, \end{cases}$$

where F is defined in (3.3), $y_a = y(a)$, where $Fy = 0$, f comes from (1.1) and Ψ is the normally referred to as the increment function of the method. Our approach appears to require the existence of an asymptotic error expansion: the local discretization error λ of any solution y to $Fy = 0$ must satisfy

$$\lambda_\nu = \phi_n(F)(\Delta_n y)_\nu = \sum_{k=1}^{\mu} (\Lambda_n e_k)_\nu h^k + (\Lambda_n g)_\nu h^{\mu+1} \quad \nu = 1, \ldots, n, \tag{3.13}$$

where $e_k \in Z^\mu, g \in Z^0$ and μ will depend on the smoothness of f (note that there is a slight abuse of notation here, λ_ν should actually be $(\lambda_n)_\nu$, however we allow this to simplify notation). If $e_k = 0$ for all $k < p$ then the numerical method ϕ is said to be consistent of order p. For one-step methods (such as

Runge-Kutta methods) where the local error can be expressed as a Butcher series, after autonomizing the original equation, there is a (formal) tree expansion

$$\phi_n(F)(\Delta_n y)_\nu = \sum_{\tau \in LT} a(\tau) F(\tau)((\Delta_n y)_{\nu-1}) \frac{h^{\rho(\tau)}}{\rho(\tau)!} \quad \nu = 1, \ldots, n. \tag{3.14}$$

Here $a(\tau)$ are computable error coefficients, $F(\tau)((\Delta_n y)_{\nu-1})$ are computable elementary differentials of f evaluated at $(\Delta_n y)_{\nu-1}$, and LT is the set of all labeled trees [9]. (Note that F used in the elementary differentials is not the same F as in the operator version of the IVP, however, this will be clear from the context.) In this case the existence of the expansion (3.13) follows easily when the grid mappings Δ_n and Λ_n correspond to an equidistant grid $\mathbb{G}_n$. It is straightforward to show that when $\mathbb{G}_n$ is equidistant (this is crucial, see Section 4.2) and if the tree expansion (3.14) holds, we have

$$\max(\|e_k\|_{Z^{\mu-k}}, \|g\|_{Z^0}) \le C_\mu \max(\|y\|_{Y^\mu}, \|y\|_{Y^\mu}^\mu), \quad k = 1, \ldots, \mu - 1 \tag{3.15}$$

where

$$C_\mu \le K \sup\{|F(\tau)(z)| : z \in [a, b] \times \mathbb{R}^d, \tau \in LT_\mu\}, \tag{3.16}$$

LT_μ is the set of all labeled trees of order $\le \mu$, and K is an integer depending on μ. We assume below that our numerical method satisfies (3.14) and that f is smooth with bounded derivatives, and hence (3.13) with (3.15) and (3.16) apply.

3.3 Stability of one-step methods

Convergence of a numerical method is usually proven in two steps. First, one proves consistency $\lambda = O(h^p)$ at each smooth solution y. Then, if the stability inequality is satisfied, one deduces convergence. Thus we seek a stability inequality

$$\|v - w\|_{Y_n^m} \le S \|\phi_n(F)v - \phi_n(F)w\|_{Z_n^m}, \qquad v, w \in Y_n^m. \tag{3.17}$$

Obviously S will depend on the norms on Y_n^m and Z_n^m, and since our norms include divided differences, it is not a surprise that S may include derivatives of f. For Runge-Kutta methods, it can be shown (see Example 4.2 of [14]) that (3.17) is valid with $S = S(\theta)$ where θ is a bound on all the partial derivatives of f of orders up to and including m and S is a nonnegative increasing function of θ. We assume below that S has this form.

4 Abstraction of the DGR scheme

The goal is to put Algorithm 2.1 into the abstract framework. Given a previously calculated solution $u \in Y_n^m$, the DGR scheme builds a continuous approximation $g(t) = \nabla_n u(t)$ to the exact solution y. The error equation for $\tilde{\delta}$

$$\tilde{\delta}'(t) = \tilde{f}(t, \tilde{\delta}(t)), \qquad \tilde{\delta}(a) = \tilde{\delta}_a = y(a) - u_0, \tag{4.1}$$

$$\tilde{f}(t, \tilde{\delta}(t)) = f(t, \tilde{\delta}(t) + g(t)) - g'(t),$$

can be put in operator form by defining $F_g : Y^m \to Z^m$, for any interpolated numerical solution g, via

$$F_g z(t) := (-z(a) + \tilde{\delta}_a, -z'(t) + f(t, z(t) + g(t)) - g'(t)). \tag{4.2}$$

The numerical scheme for $\tilde{\delta}$ is then

$$\phi_n(F_g)\delta = 0, \qquad g = \nabla_n u(t).$$

In particular, if ϕ is the implicit midpoint rule we get that

$$\phi_n(F_g)(u)_\nu = \begin{cases} -u_0 + \tilde{\delta}_a & \nu = 0 \\ -\frac{u_\nu - u_{\nu-1}}{h} + f\left(\hat{t}_\nu, \frac{u_\nu + u_{\nu-1}}{2} + g(\hat{t}_\nu)\right) + g'(\hat{t}_\nu) & \nu = 1, \ldots, n, \end{cases}$$

where $\hat{t}_\nu = (t_{\nu-1} + t_\nu)/2$. Before we can prove the desired convergence of the DGR method we need to make some adjustments. In particular the spaces Y_n^m and Z_n^m must be altered slightly before we can start proving any theorems, and that is the theme of the next section.

4.1 Theoretical adjustments

We now adjust the previous theoretical framework so that convergence can be proven for the DGR scheme. In this section we consider the IVP (1.1) on the interval $[0, T]$. The previously defined spaces and norms interfere with the boundedness of the Lagrange interpolation operator $\nabla_n : Y_n^m \to Y^m$. Recall that we are considering the sequence $\{u_n\}_{n\in\mathbb{N}}$ for $u_n \in Y_n^m$. Obtaining a polynomial $g(t) = \nabla_n u_n(t)$ of arbitrarily high degree as $h \to 0$ does not make any sense, so computationally we only interpolate a fixed finite number of points. But piecewise interpolation may introduce unsmoothness so boundedness of ∇_n is impossible in spaces Y^m with $m > 0$. Thus we consider each correction interval as a separate IVP. In other words split the interval $[0, T]$ into N subintervals $[T_{k-1}, T_k]$, $k = 1, \ldots, N$ where $T_0 = 0$, $T_N = T$ and $[0, T] = \cup_{k=1}^N [T_{k-1}, T_k]$. Let $n \in \mathbb{N}$ and for each k let $\mathbb{G}_n^k$ be an equidistant grid on $[T_{k-1}, T_k]$ with $h_k = (T_k - T_{k-1})/n$. Letting $h = \max_k(T_k - T_{k-1})$ we require that there is a constant C (independent of N) such that

$$h \leq C \min_{1\leq k\leq N} h_k. \tag{4.3}$$

Now define the grid $\mathbb{G}_{Nn}$ on $[0, T]$ by $\mathbb{G}_{Nn} = \cup_{k=1}^N \mathbb{G}_n^k$ such that $T_k = t_{kn} \in \mathbb{G}_{Nn}$. We can now define the vector space Y_{Nn}^m according to $\mathbb{G}_{Nn}$. In particular, let

$$Y_{Nn}^m := \mathbb{R}^{d(Nn+1)},$$

where we interpret $u \in Y_{Nn}^m$ as

$$u = (u_0, \ldots, u_{Nn})^T, \qquad u_\nu \in \mathbb{R}^d,$$

where u_ν corresponds to $t_\nu \in \mathbb{G}_{Nn}$. Also, for $k = 1, \ldots, N$, let $u^k \in \mathbb{R}^{d(n+1)}$ be defined by

$$u^k := (u_{(k-1)n}, \ldots, u_{kn})^T.$$

We will for simplicity use the notation $u^k = (u_0^k, \ldots, u_n^k)^T$ below. We can now define the norm on Y_{Nn}^m by

$$\|u\|_{Y_{Nn}^m} := \max_{1\leq k\leq N} \{\|u^k\|_{Y_n^m}\}.$$

Similarly let

$$Z_{Nn}^m := \mathbb{R}^{d(Nn+1)}, \qquad \|z\|_{Z_{Nn}^m} := \max_{1\leq k\leq N} \{\|z^k\|_{Z_n^m}\}.$$

These norms do not require smoothness across subgrid boundaries, and are therefore convenient for the analysis of deferred correction schemes which work on one subinterval at a time. We now split the IVP into N IVPs. In particular, we have the IVPs

$$F_k y = 0, \qquad k = 1, \ldots, N,$$

$$F_k z(t) := \left(-z(t_{(k-1)n}) + y(t_{(k-1)n}), -z'(t) + f(t, z(t))\right), \qquad t \in [T_{k-1}, T_k], \tag{4.4}$$

where y satisfy $Fy = 0$ and F is defined in (3.3). Suppose $u \in Y_{Nn}^m$ is an approximation to the discretization $\Delta_{Nn} y$ of the exact solution y to the IVP $Fy = 0$. Let $g_k = \nabla_n u^k$ be the local Lagrange interpolant to the grid values on the k-th subgrid. The correction equations require $\delta \in Y_{Nn}^m$ to satisfy

$$\begin{aligned} \phi_n(F_{g_k})(\delta^k)_\nu &= \begin{cases} -\delta_0^k + \delta_n^{k-1} & k = 1, \dots, N \\ -\frac{\delta_\nu^k - \delta_{\nu-1}^k}{h} + \Psi(h, \delta_{\nu-1}^k, \tilde{f}_{g_k}) & \nu = 1, \dots, n, \quad k = 1, \dots, N \end{cases} \\ &= 0 \end{aligned} \tag{4.5}$$

where, for $\tilde{\delta} \in Y^m$, $\tilde{f}_{g_k}(t, \tilde{\delta}) = f(t, \tilde{\delta} + g_k(t)) + g_k'(t)$ and $\delta_0^1 = 0$, assuming exact starting value. With this framework, we can state the main lemma of the convergence proof:

Lemma 4.1. *Let y satisfy $Fy = 0$, where F is defined in (3.3) and $f \in C^\infty(\mathbb{R}^{d+1})$ with bounded derivatives. Let $n \in \mathbb{N}$ and $\mathbb{G}_n$ be an equidistant grid on $[a, b]$ with stepsize h. Suppose that the numerical method ϕ is stable (with stability properties as in Section 3.3) and consistent of order p with the local discretization error satisfying (3.13) and (3.15). Suppose also that for integers m, p, r with $n \geq m + p + r$, a numerical solution $u \in Y_n^{m+p}$ satisfies the error bound $\|u - \Delta_n y\|_{Y_n^{m+p}} \leq Ch^r$ (for some $C > 0$). Then the correction δ satisfies the order-$(r + p)$ error bound*

$$\|\delta + u - \Delta_n y\|_{Y_n^m} \leq C_n \max\{\|\delta_0 + u_0 - (\Delta_n y)_0\|_{\mathbb{R}^d}, h^{r+p}\}$$

(where $\|\cdot\|_{\mathbb{R}^d}$ denotes the max norm) for some C_n depending on n, $[a, b]$ and f (and its derivatives). Here δ is the correction satisfying $\phi_n(F_{\nabla_n u})\delta = 0$ with initial value δ_0 and where $F_{\nabla_n u}$ is defined in (4.2). Moreover, the statement is valid (with the same C_n) for any interval $[\tilde{a}, \tilde{b}] \subset [a, b]$.

Proof. Throughout the proof C may represent various constants. Let $g(t) = \nabla_n u(t)$. By the stability assumption, the assumption that f has bounded derivatives and the definition of δ we have

$$\begin{aligned} \|\delta + u - \Delta_n y\|_{Y_n^m} &\leq S(\theta)\|\phi_n(F_g)\delta - \phi_n(F_g)(\Delta_n y - u)\|_{Z_n^m} \\ &= S(\theta)\|\phi_n(F_g)(\Delta_n y - u)\|_{Z_n^m}, \end{aligned} \tag{4.6}$$

for some $S(\theta) > 0$ independent of n, where S is an increasing function of θ and θ is a bound on the partial derivatives of $[a, b] \times \mathbb{R}^d \ni (t, z) \mapsto f(t, z + \nabla_n u(t)) - \nabla_n u'(t)$ up to order m. Also, recall that F_g is defined in (4.2). We will discuss the dependency on θ below. Since the exact solution of the correction equation is $\tilde{\delta} = y - \nabla_n u$ and $\Delta_n \nabla_n$ is the identity, it follows that $\phi_n(F_g)(\Delta_n \tilde{\delta}) = \phi_n(F_g)(\Delta_n y - u)$ is the local discretization error of the correction IVP $F_g \tilde{\delta} = 0$. Since ϕ is a scheme that is consistent of order p, with an asymptotic error expansion as in (3.13), it follows that

$$\phi_n(F_g)(\Delta_n y - u)_\nu = \begin{cases} \delta_0 + u_0 - (\Delta_n y)_0 & \nu = 0 \\ \sum_{j=0}^m h^{p+j}(\Lambda_n e_{p+j})_\nu + h^{m+p+1}(\Lambda_n g)_\nu & \nu = 1, \dots, n. \end{cases} \tag{4.7}$$

Thus, the local discretization error of the corrected solution satisfies

$$\begin{aligned} &\|\phi_n(F_g)(\Delta_n y - u)\|_{Z_n^m} \\ &\leq \max\left(\|\delta_0 + u_0 - (\Delta_n y)_0\|_{\mathbb{R}^d}, \sum_{j=0}^m h^{p+j}\|\Lambda_n e_{p+j}\|_{Z_n^m} + h^{m+p+1}\|\Lambda_n g\|_{Z_n^m}\right), \end{aligned}$$

where the norm $\|\cdot\|_{\mathbb{R}^d}$ is the max norm. We will now consider the last part of this bound in the following claim.

Claim: There is a C depending on n and $[a,b]$ such that

$$\sum_{j=0}^{m} h^{p+j}\|\Lambda_n e_{p+j}\|_{Z_n^m} + h^{m+p+1}\|\Lambda_n g\|_{Z_n^m} \leq Ch^{r+p}.$$

Moreover, the claim is valid (with the same C) for any interval $[\tilde{a},\tilde{b}] \subset [a,b]$. To prove the claim, note that by the extended mean value theorem for higher order derivatives and divided differences, the operator norm of the point evaluation Λ_n satisfies $\|\Lambda_n\|_{Z^k \to Z_n^{k+l}} \leq Mh^{-l}$ for any positive integers k, l and a constant $M > 0$ depending on l, n and $[a,b]$, however independent of any $[\tilde{a},\tilde{b}] \subset [a,b]$. Thus

$$\begin{aligned}\sum_{j=0}^{m} h^{p+j}\|\Lambda_n e_{p+j}\|_{Z_n^m} + h^{m+p+1}\|\Lambda_n g\|_{Z_n^m} \\ \leq h^p \sum_{j=0}^{m} M_j \|e_{p+j}\|_{Z^{m-j}} + h^{p+1} M_m \|g\|_{Z^0}\end{aligned}$$

where the constants M_j may depend on n and $[a,b]$ however independent of any $[\tilde{a},\tilde{b}] \subset [a,b]$. By the asymptotic error expansion assumptions (3.15) and (3.16), $\|e_{p+j}\|_{Z^{m-j}}$ (for $0 \leq j \leq m$) and $\|g\|_{Z^0}$ are bounded by

$$\widetilde{C} \max(\|\nabla_n u - y\|_{Y^{m+p}}, \|\nabla_n u - y\|_{Y^{m+p}}^{m+p})$$

where

$$\widetilde{C} \leq K \sup\{|F(\tau)(z)| : z \in [a,b] \times \mathbb{R}^N, \tau \in LT_{m+p}\},$$

with $F(\tau)(z)$ an elementary differential of the autonomized correction equation

$$\tilde{\delta}'(t) = f(t, \tilde{\delta}(t) + \nabla_n u(t)) - \nabla_n u'(t), \qquad t \in [a,b],$$

and K is an integer depending on $m+p$. Thus, our claim will follow if we can bound $\widetilde{C}$ and show that there is a $C > 0$ such that $\|\nabla_n u - y\|_{Y^{m+p}} \leq Ch^r$. Let us start with the latter. Note that the triangle inequality gives

$$\|\nabla_n u - y\|_{Y^{m+p}} \leq C\|u - \Delta_n y\|_{Y_n^{m+p}} + \|\nabla_n \Delta_n y - y\|_{Y^{m+p}},$$

where C (depending on n and $[a,b]$, however independent of any $[\tilde{a},\tilde{b}] \subset [a,b]$) is a bound on the operator norm of Lagrange interpolation $\|\nabla_n\|$ (recall Lemma 3.1). Interpolation and point evaluation are related by $\|\nabla_n \Delta_n y - y\|_{Y^{m+p}} \leq Ch^{n-m+p}$ (C may depend on n and $[a,b]$, however independent of any $[\tilde{a},\tilde{b}] \subset [a,b]$. Thus, by the assumption that $\|u - \Delta_n y\|_{Y_n^{m+p}} \leq Ch^r$ and the choice of n (recall that $n \geq m+p+r$), it follows that $\|\nabla_n u - y\|_{Y^{m+p}} \leq Ch^r$. Where C depends on n and $[a,b]$, however independent of any $[\tilde{a},\tilde{b}] \subset [a,b]$.

Now returning to the bound on $\widetilde{C}$. By the definition of elementary differentials, $\widetilde{C}$ will depend only on f (and its derivatives) and n if $\|\nabla_n u\|_{Y^k}$ is bounded for $k = p+m+1$. To bound $\|\nabla_n u\|_{Y^k}$ we observe that, for any positive integer l the identity operator I is bounded by $M_l h^{-l}$ (for some $M_l > 0$) from Y_n^m to Y_n^{m+l}. Thus

$$\|u - \Delta_n y\|_{Y_n^k} \leq Ch^{-1}\|u - \Delta_n y\|_{Y_n^{m+p}}.$$

Hence, it follows that

$$\begin{aligned}\|\nabla_n u\|_{Y^k} &\leq \|\nabla_n u - y\|_{Y^k} + \|y\|_{Y^k} \\ &\leq Ch^{r-1} + \|y\|_{Y^k},\end{aligned} \tag{4.8}$$

by the assumption that $\|u - \Delta_n y\|_{Y_n^{m+p}} \leq Ch^r$, and the claim is proved.

Note that the lemma will follow by (4.6) and the claim if we can control θ from (4.6). Recall that θ is a bound on the partial derivatives of $[a,b]\times\mathbb{R}^d \ni (t,z) \mapsto f(t,z+\nabla_n u(t)) - \nabla_n u'(t)$ up to order m. Since all partial derivatives of f are bounded, it suffices to bound $\|\nabla_n u\|_{Y^{m+1}}$ which follows from (4.8). □

Our main theorem follows immediately from the main lemma:

Theorem 4.2. *Let y satisfy $Fy = 0$ (where F is as in Lemma 4.1 except with the interval $[0,T]$) and suppose that ϕ satisfies the assumptions in Lemma 4.1. Let, for $n, N \in \mathbb{N}$, $\mathbb{G}_{Nn}$ be a grid on $[0,T]$ with properties as in Section 4.1. Suppose that a numerical solution $u \in Y_{Nn}^{m+p}$ satisfies $\|u - \Delta_{Nn} y\|_{Y_{Nn}^{m+p}} \leq Ch^r$ and $\|u_0 - (\Delta_{Nn} y)_0\|_{\mathbb{R}^d} \leq Ch^{r+p}$ (the max norm) for some $C > 0$ then the corrected solution $u + \delta$ satisfies*

$$\|\delta + u - \Delta_{Nn} y\|_{Y_{Nn}^m} \leq C_n h^{p+r}$$

(where C_n depends only on n and not on N) whenever δ is determined by (4.5).

Proof. Apply Lemma 4.1 to the subproblem $F_k y = 0$ defined in (4.4) and the subgrid correction δ^k satisfying (4.5) to get for $1 < k \leq N$ that

$$\|\delta^k + u^k - (\Delta_n y)^k\|_{Y_n^m} \leq C_n \max\left(|(\delta_0^{k-1} + u_0^{k-1} - (\Delta_n y)_0^{k-1})|, h_k^{p+r}\right),$$

(recall the notation from Section 4.1) and

$$\|\delta^1 + u^1 - (\Delta_n y)^1\|_{Y_n^m} \leq C_n \max\left(\|u_0 - (\Delta_{Nn} y)_0\|_{\mathbb{R}^d}, h_1^{p+r}\right)$$

where C_n depends only on n (and not on N) The theorem follows by the definition of the global norm $\|\cdot\|_{Y_{Nn}^m}$, the assumption (4.3) that

$$h \leq C \min_{1\leq k\leq N} h_k,$$

the assumption that $\|u_0 - (\Delta_{Nn} y)_0\|_{\mathbb{R}^d} \leq Ch^{r+p}$ and induction on k. □

Remark 4.3. The assumption that f is smooth with bounded derivatives may seem like a strong assumption. We do this simply to simplify the exposition. The assumption that f has bounded derivatives may just as easily be replaced with the assumption that f has bounded derivatives on compact sets. Also, the assumption that f is infinitely smooth can easily be replaced by the assumption that f is sufficiently smooth (i.e. having enough derivatives required in the proof).

4.2 Error Expansions and Equidistant Grids

Note that the assumption in Theorem 4.2 that $\mathbb{G}_{Nn} = \cup_{k=1}^N \mathbb{G}_n^k$, where $\mathbb{G}_n^k$ is equidistant is absolutely crucial. We use this fact already in Lemma 3.1, however, this is really only for convenience. In fact, one could actually carry out the proof of Theorem 4.2 without the assumption of equidistant subgrids. However, the increase in order when correcting may not happen with non-equidistant subgrids (see Section 5). Thus, one may ask: where is the requirement of equidistant subgrids hidden? The answer is: it is hidden in the assumption in Lemma 4.1 that there exists an expansion of the local error of the following form

$$\phi_n(F)(\Delta_n y)_\nu = \sum_{k=1}^{\mu} (\Lambda_n e_k)_\nu h^k + (\Lambda_n g)_\nu h^{\mu+1} \quad \nu = 1,\dots,n, \tag{4.9}$$

where $e_k \in Z^\mu, g \in Z^0$ and μ will depend on the smoothness of f. And also that we have

$$\max(\|e_k\|_{Z^{\mu-k}}, \|g\|_{Z^0}) \leq C_\mu \max(\|y\|_{Y^\mu}, \|y\|_{Y^\mu}^\mu), \quad k = 1,\dots,\mu-1. \tag{4.10}$$

The problem is that (4.9) with (4.10) may not be true (even Euler's method will not work) on a non-equdistant grid. In particular, the functions $\{e_k\}_{k=1}^{\mu}$ may not exist. To see this consider the following example: Suppose we want to solve the equation

$$y'(t) = y(t), \qquad y(0) = y_0 = 1, \quad t \in [0, 2h], \tag{4.11}$$

for some $h > 0$. Let us, for $a \in (0, 1]$, define the grid $\mathbb{G}_2 = \{0, ah, (2-a)h\}$ on $[0, 2h]$ (with corresponding grid maps Δ_2 and Λ_2). Let ϕ denote Euler's method. (If we use Euler's method twice in the DGR method with $\mathbb{G}_2$ and solve (4.11) we get order one when $a \neq 1$ and order two when $a = 1$.) Note that

$$\phi_2(F)(\Delta_2 y)_\nu = \begin{cases} -\frac{e^{ah}-1}{ah} + 1 & \nu = 1 \\ -\frac{e^{2h}-e^{ah}}{(2-a)h} + e^{ah} & \nu = 2. \end{cases}$$

Thus, after applying some easy calculation with series expansions of the exponential function we obtain that

$$\phi_2(F)(\Delta_2 y)_\nu = \begin{cases} -\left(\frac{ah}{2!} + \mathcal{O}(h^2)\right) & \nu = 1 \\ -\left(\frac{(2-a)h}{2!} + \mathcal{O}(h^2)\right) & \nu = 2. \end{cases}$$

Now let $\mu = 3$ in (4.9) (this is the value we need if we were to use the DGR scheme and Euler's method to obtain a second order numerical solution). Now suppose that $e_1 \in Z^2$ exists as in the expansion (4.9) with the property (4.10). By (4.9) we get that

$$e_1(0) = -\frac{a}{2}, \qquad e_1(ah) = \frac{a-2}{2}.$$

Thus, by the mean value theorem, we have that

$$\|e_1\|_{Z^2} \geq \frac{1-a}{ah} \to \infty, \quad h \to 0, \quad \forall\, a \in (0, 1).$$

In particular, such an e_1 satisfying (4.10) can only exist when $a = 1$, namely, when the grid $\mathbb{G}_2$ is equidistant. Another assumption that may require equidistant sub grids is the assumption in Lemma 4.1 that

$$\|u - \Delta_n y\|_{Y_n^{m+p}} \leq Ch^r, \tag{4.12}$$

where u is the original approximation (that we are trying to improve) to $\Delta_n y$. However, if we consider (4.11) and suppose that we have an approximation $u = \{u_0, u_1, u_2\}$ to $\Delta_2 y$ such that u is produced by Euler's method (note that $u_0 = y_0$), then a short calculation actually yields

$$\|u - \Delta_2 y\|_{Y_2^1} \leq Ch, \qquad \forall\, a \in (0, 1], \tag{4.13}$$

which is what we need to apply Lemma 4.1 to be able to correct with Euler's method and yield a second order numerical approximation. To see this note that

$$\|u - \Delta_2 y\|_{Y_2^1} = \max\{|u_\nu - (\Delta_2 y)_\nu|, |Du_\mu - D(\Delta_2 y)_\mu| : \nu = 1, 2, 3,\ \mu = 1, 2\}.$$

It is clear that

$$\max\{|u_\nu - (\Delta_2 y)_\nu| : \nu = 1, 2, 3\} = \mathcal{O}(h), \quad h \to 0,\ \forall\, a \in (0, 1].$$

Also, note that since we are considering (4.11) and u is produced by Euler's method, it follows that

$$Du_1 = u_0, \quad Du_2 = u_1, \quad D(\Delta_2 y)_1 = \frac{y_0(e^{ah} - 1)}{ah}, \quad D(\Delta_2 y)_2 = \frac{y_0(e^{2h} - e^{ah})}{(2-a)h}.$$

Hence,

$$Du_1 - D(\Delta_2 y)_1 = y_0 \frac{ah}{2} + \mathcal{O}(h^2), \qquad Du_2 - D(\Delta_2 y)_2 = y_0 \left(ah - \frac{2+a}{2} h \right) + \mathcal{O}(h^2).$$

And hence we get (4.13). In particular, it is the problem with the existence of (4.9) with (4.10) and not (4.12) that is the issue. Having said that, of course on a nonlinear problem (4.12) may be an issue on a non-equidistant grid. Note, however, that if ϕ is a numerical method, that satisfies (4.9) with (4.10) and u is the numerical solution corresponding to ϕ, then (4.12) will be satisfied (as long as f is sufficiently smooth). Indeed, this can be seen by Lemma 4.1 by interpreting u as being obtained from the DGR method correcting from the "zero numerical solution". In this case we need $\|\Delta_n y\|_{Y_n^{m+2p}} \leq C$, and this will be satisfied given sufficient smoothness.

5 Numerical examples

In this section we will test the method on a standard example; the Van der Pol equation. The equation is given by

$$x''(t) + \mu(1 - x(t)^2)x'(t) + x(t) = 0, \qquad t \in [0, T].$$

Making the usual transformation, $y_1(t) = x(t)$, $y_2(t) = \mu x'(t)$, and $t = t/\mu$ yields the system of equations

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= (-y_1(t) + (1 - y_1(t)^2)y_2(t))/\epsilon, \qquad \epsilon = 1/\mu^2, \qquad t \in [0, T]. \end{aligned} \tag{5.1}$$

5.1 Example with Euler's Method

As a first example it is natural to use a basic first order one-step scheme. In particular, we have tested the DGR method on (5.1) with the Euler method, thus

$$\phi_n(F)(u)_\nu = \begin{cases} -u_0 + y_0 & \nu = 0 \\ -\frac{u_\nu - u_{\nu-1}}{h} + f(t_{\nu-1}, u_{\nu-1}) & \nu = 1, \ldots, n. \end{cases}$$

We consider (5.1) with $\mu = 1$, $T = 6$ and $y_0 = y(0) = [2, 2/3]^T$. The "exact" solution has been computed with MATLAB's ode45 with 'AbsTol' = 10^{-16} and 'RelTol' = 10^{-16}. Table 5.1 shows error $Err = [er_1, er_2, er_3, er_4]$ at $T = 6$ (upper part), where er_j corresponds to $\|y(T) - y_{\text{comp},h_j}\|_2$ where y_{comp,h_j} is the computed value at T according to the step size $h_j = T/N_j$ $j = 1, \ldots, 4$, as well as the measured order $\log(er_j/er_{j+1})/\log(2)$ (lower part). In all examples, each interval $[(k-1)h_j, kh_j]$, $k = 1, \ldots N_j$ has been divided into $n = 7$ subintervals used in the interpolation according to Section 4.1. The notation $\text{DGR}^k_{\text{Eul}}$ denotes k corrections with Euler's method from the zero solution and hence the $\text{DGR}^k_{\text{Eul}}$ method should be of order k.

5.2 Example with 2-order Runge-Kutta Method

In this example we will test the DGR method on (5.1) with the midpoint method (which is of order 2), thus

$$\phi_n(F)(u)_\nu = \begin{cases} -u_0 + y_0 & \nu = 0 \\ -\frac{u_\nu - u_{\nu-1}}{h} + f(t_{\nu-1} + \frac{h}{2}, u_{\nu-1} + \frac{h}{2} f(t_{\nu-1}, u_{\nu-1})) & \nu = 1, \ldots, n. \end{cases}$$

We consider (5.1) with the same data as in the previous example. And the tested results have been collected in Table 5.2 similar to Table 5.1. In all examples, each interval $[(k-1)h_j, kh_j]$, $k = 1, \ldots N_j$ has been

N	h	Euler	$\mathrm{DGR}^2_{\mathrm{Eul}}$	$\mathrm{DGR}^3_{\mathrm{Eul}}$	$\mathrm{DGR}^4_{\mathrm{Eul}}$	$\mathrm{DGR}^5_{\mathrm{Eul}}$	$\mathrm{DGR}^6_{\mathrm{Eul}}$	$\mathrm{DGR}^7_{\mathrm{Eul}}$
12	0.5000	7.78e-01	2.96e-02	3.76e-03	4.49e-03	2.81e-03	2.01e-03	5.72e-04
24	0.2500	3.67e-01	9.12e-03	6.93e-04	2.49e-05	2.35e-05	4.30e-06	2.42e-06
48	0.1250	1.78e-01	2.29e-03	9.10e-05	1.94e-06	8.76e-07	4.16e-08	2.03e-08
96	0.0625	8.50e-02	5.80e-04	1.15e-05	1.28e-07	2.90e-08	5.60e-10	1.45e-10
12	0.5000							
24	0.2500	1.08	1.69	2.44	7.49	6.91	8.87	7.88
48	0.1250	1.07	1.99	2.93	3.68	4.74	6.69	6.90
96	0.0625	1.04	1.98	2.98	3.92	4.92	6.25	7.12

Table 5.1: The table shows error and measured order of the DGR method using the Euler scheme.

N	h	RK2	$\mathrm{DGR}^2_{\mathrm{RK2}}$	$\mathrm{DGR}^3_{\mathrm{RK2}}$	$\mathrm{DGR}^4_{\mathrm{RK2}}$	$\mathrm{DGR}^5_{\mathrm{RK2}}$	$\mathrm{DGR}^6_{\mathrm{RK2}}$	$\mathrm{DGR}^6_{\mathrm{RK2,ne}}$
3	2.00	2.87e-02	1.72e-01	3.57e-01	2.29e-01	2.71e-01	2.82e-01	8.82e-04
6	1.00	9.67e-03	7.84e-05	1.07e-05	7.76e-06	9.00e-06	8.87e-06	1.28e-06
12	0.50	2.67e-03	7.33e-06	2.92e-08	2.43e-09	2.17e-09	2.18e-09	2.85e-09
24	0.25	6.94e-04	5.61e-07	1.99e-10	1.790e-12	6.06e-13	1.70e-13	3.44e-11
3	2.00							
6	1.00	1.57	11.10	15.02	14.85	14.88	14.96	9.42
12	0.50	1.86	3.42	8.52	11.64	12.02	11.99	8.81
24	0.25	1.94	3.71	7.20	10.41	11.80	13.64	6.38

Table 5.2: The table shows error and measured order of the DGR method using the midpoint scheme.

divided into $n = 14$ subintervals used in the interpolation according to Section 4.1. The notation $\mathrm{DGR}^k_{\mathrm{RK2}}$ denotes k corrections with the midpoint method (we use RK2 for short) from the zero solution and hence the $\mathrm{DGR}^k_{\mathrm{RK2}}$ method should be of order $2k$. Note that to show that the correction scheme does not work with a non-equidistant sub grid we have plotted the error and measured order of 6 corrections with a RK2 on a non-equidistant grid . This method has been denoted by $\mathrm{DGR}^6_{\mathrm{RK2,ne}}$.

5.3 Example with Mixed Methods

In this example we have tested the DGR method on (5.1) with mixed methods, meaning both the Euler method as well as RK2 in the corrections. We consider (5.1) with the same data as in the previous examples. And the tested results have been collected in Table 5.3 similar to Table 5.1. In all examples, each interval $[(k-1)h_j, kh_j]$, $k = 1, \dots N_j$ has been divided into $n = 10$ subintervals used in the interpolation according to Section 4.1. The notation $\mathrm{DGR}^{j,k}_{\mathrm{Eu,R2}}$ means j corrections with the Euler method first and then k corrections with RK2. Similarly, $\mathrm{DGR}^{j,k}_{\mathrm{R2,Eu}}$ means means j corrections with RK2 first and then k corrections with the Euler method. Thus, $\mathrm{DGR}^{j,k}_{\mathrm{Eu,R2}}$ should be of order $j + 2k$ and $\mathrm{DGR}^{j,k}_{\mathrm{R2,Eu}}$ should be of order $2j + k$.

Note that in Table 5.2 and Table 5.3 the grid is rather coarse, because observing higher orders is otherwise not possible in double precsion. On the other hand, the observed orders are not yet in the "asymptotic regime" meaning that the step size h may not be close enough to zero to actually reveal the actual order. This is why one can observe a slightly more optimistic value of the measured order than what is theoretically predicted.

N	h	$\mathrm{DGR}^{2,3}_{\mathrm{Eu,R2}}$	$\mathrm{DGR}^{3,2}_{\mathrm{R2,Eu}}$	$\mathrm{DGR}^{5,2}_{\mathrm{Eu,R2}}$	$\mathrm{DGR}^{2,5}_{\mathrm{R2,Eu}}$	$\mathrm{DGR}^{7,1}_{\mathrm{Eu,R2}}$	$\mathrm{DGR}^{4,1}_{\mathrm{R2,Eu}}$
6	1.000	2.25e-04	1.92e-03	2.85e-03	7.23e-03	1.25e-02	8.86e-05
12	0.500	1.28e-06	9.73e-06	2.57e-06	2.26e-06	8.24e-06	2.21e-06
24	0.250	4.09e-10	5.75e-09	3.86e-10	5.85e-09	4.30e-10	2.67e-09
48	0.125	7.03e-13	1.60e-12	1.64e-13	3.10e-12	8.17e-13	7.20e-13
6	1.000						
12	0.500	7.46	7.62	10.12	11.64	14.88	5.33
24	0.250	11.61	10.72	12.70	8.59	12.02	9.69
48	0.125	9.18	11.82	11.20	10.88	11.80	11.86

Table 5.3: The table shows error and measured order of the DGR method using mixed methods.

5.4 Example with Multistep Methods

Even though our convergence proof only covers one-step methods, we could not resist the temptation to experiment with multistep methods. We have tested the DGR method on (5.1) with mixed methods, both one-step and multistep. The test has been carried out as in the previous examples, and the results are displayed in Table 5.4 similar to Table 5.1. The notation $\mathrm{DGR}^{\mathrm{Eu}}_{\mathrm{ABj}}$ means that a solution has been computed with a j-step (order j) Adams-Bashford method and then corrected (according to the framework in Section 4.1) with Euler's method. Also, $\mathrm{DGR}^{\mathrm{R2}}_{\mathrm{ABj}}$ means that a solution has been computed with a j-step (order j) Adams-Bashford method and then corrected with a Runge-Kutta method of order two. In all examples, each interval $[(k-1)h_j, kh_j]$, $k = 1, \ldots N_j$ has been divided into $n = 8$ subintervals. Note that the correction with Euler's method improves the order of the method with one, however, the correction with a second order RK method does not improve the order of the method with two but rather one. We do not want to speculate why this is the case, but leave this for future research. Although correcting with a higher order method does not seem to be a very good idea, using correction with Euler's method for easy error control may be worth looking into.

The notation $\mathrm{DGR}^{\mathrm{ABj}}_{\mathrm{R4}}$ means that a solution has been computed with a Runge-Kutta method of order four and then this solution has been corrected with an Adams-Bashford method of order j. Note that in this case, it seems from the experiment, that the total order of the method is order of Runge-Kutta + order of multistep. Note that a DGR method based on a mix between one-step and multistep may have a connection to the General Linear Methods of Butcher [4, 5]. We postpone the theoretical analysis of the numerical results to future papers.

6 Conclusion

We have proved convergence for the new DGR method based on one-step methods, and as the numerical results suggest, this is a nice new addition to the deferred correction family. Since there previously did not exist any convergence proof for this method, new theoretical tools had to be built. However, as the numerical examples with multistep methods suggest, the convergence theory is not fully understood. We strongly believe that our analysis, based on smooth local error estimates, should be extendable to multistep methods.

The most notably advantages of the method is that it is easy to build high order schemes from low order methods, however, this feature is general for deferred correction methods. What makes this method really stand out is the advantage that one may be able to at least have an easy error control for multistep methods, and also build methods from one-step methods and multistep methods. The connection to Butcher's General

N	h	$\mathrm{DGR}^{\mathrm{Eu}}_{\mathrm{AB3}}$	$\mathrm{DGR}^{\mathrm{Eu}}_{\mathrm{AB4}}$	$\mathrm{DGR}^{\mathrm{R2}}_{\mathrm{AB4}}$	$\mathrm{DGR}^{\mathrm{AB3}}_{\mathrm{Eu}}$	$\mathrm{DGR}^{\mathrm{AB2}}_{\mathrm{R4}}$	$\mathrm{DGR}^{\mathrm{AB3}}_{\mathrm{R4}}$
10	0.60	1.45e-02	8.96e-03	6.28e-05	2.15e-02	1.41e-04	1.83e-05
20	0.30	9.15e-04	2.20e-04	1.80e-05	3.43e-04	2.89e-07	2.34e-07
40	0.15	5.72e-05	5.94e-06	5.90e-07	1.71e-05	9.83e-10	1.18e-10
80	0.075	3.59e-06	1.69e-07	1.73e-08	9.41e-07	1.61e-11	3.41e-13
10	0.60						
20	0.30	3..99	5.35	1.81	5.97	8.93	5.33
40	0.15	4.00	5.21	4.93	4.33	8.20	9.69
80	0.075	4.00	5.13	5.09	4.18	5.93	11.86

Table 5.4: The table shows error and measured order of the DGR method using one-step and multistep methods.

Linear Methods [4, 5] is not obvious, but there is a chance that what one eventually ends up with is a way of generating General Linear Methods. We postpone such analysis to future papers.

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