# Theoretical Physics 1
# Answers to Examination 2001

Warning — these answers have been completely retyped... Please report any typos/errors.
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Q1. Bookwork: Hamilton's principle is $\delta \int \mathrm{d}t \; L(q_i, \dot{q}_i, t) = 0$ and leads (via the calculus of variations) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_i} \tag{1}$$

i.e. $N$ 2nd-order equations for the coordinates $q_i$.

The kinetic energy of the masses at $B, B'$ is $2 \times \frac{1}{2} m_1 a^2(\Omega^2 \sin^2\theta \; + \dot{\theta}^2)$. The mass at $A'$ has velocity $2a\dot{\theta}\sin\theta$ so contributes $2m_2 a^2 \sin^2\theta \; \dot{\theta}^2$. The potential energy is $V = -ga\cos\theta(2m_1 + 2m_2)$ so the Lagrangian $L = T - V$ is

$$L = m_1 a^2(\Omega^2 \sin^2\theta + \dot{\theta}^2) + 2m_2 a^2 \sin^2\theta \; \dot{\theta}^2 + 2ag\cos\theta(m_1 + m_2) \; . \tag{2}$$

The conjugate momentum is $p_\theta = \partial L/\partial\dot{\theta} = a^2\dot{\theta}(2m_1 + 4m_2\sin^2\theta)$. The equation of motion is (note a partial cancellation in the $\dot{\theta}^2$ term)

$$a^2(2m_1 + 4m_2\sin^2\theta)\ddot{\theta} + 4a^2 m_2 \sin\theta\cos\theta \; \dot{\theta}^2 = 2a\sin\theta\left(m_1 a\Omega^2\cos\theta - g(m_1 + m_2)\right) \; . \tag{3}$$

In equilibrium the LHS is zero so

$$\cos\theta_0 = \frac{g(m_1 + m_2)}{m_1 a\Omega^2} \; . \tag{4}$$

The stable position has to be $\theta = 0$ unless $\cos\theta_0 \leq 1$, so the critical condition is $\Omega^2 = g(m_1 + m_2)/m_1 a$.

For small oscillations we ignore the $\dot{\theta}^2$ term and expand the RHS, getting

$$a(m_1 + 2m_2\sin^2\theta_0) \; \ddot{\theta} \approx \delta\theta\left(m_1 a\Omega^2(\cos^2\theta_0 - \sin^2\theta_0) - g(m_1 + m_2)\cos\theta_0\right) \; , \tag{5}$$

where $\delta\theta \equiv \theta - \theta_0$. Substituting $g(m_1 + m_2)$ from (4) above, the $\cos^2$ term disappears and we get

$$a(m_1 + 2m_2\sin^2\theta_0)\ddot{\theta} \approx -\delta\theta m_1 a\Omega^2 \sin^2\theta_0 \; . \tag{6}$$

The angular velocity of small oscillations is thus $\Omega\sin\theta_0/\sqrt{1 + 2(m_2/m_1)\sin^2\theta_0}$.

Give full marks for any reasonable expression.

Q2. Bookwork: the conjugate momenta are $p_i \equiv \partial L/\partial q_i$. The Hamiltonian is

$$H \equiv \sum_i p_i \dot{q}_i - L \; , \tag{7}$$

which is a function of $(q_i, p_i)$ but not $\dot{q}_i$. Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \; ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \; , \tag{8}$$

i.e. a set of $2N$ first-order equations for the coordinates and momenta.

The Lagrangian is

$$L = \tfrac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \; \dot{\phi}^2 \right) - V(r) \tag{9}$$

The momenta are

$$p_r = m\dot{r} \; ; \quad p_\theta = mr^2 \dot{\theta} \; ; \quad p_\phi = mr^2 \sin^2\theta \; \dot{\phi} \tag{10}$$

and the Hamiltonian (which must be expressed in terms of $(q_i, p_i)$) is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + V(r) \; . \tag{11}$$

To show that $p_\phi$ is constant note that $\partial H / \partial \phi = 0$ (but method using Lagrangian symmetry or any other valid approach gets full marks).

From Hamilton's equations we have

$$\dot{p}_\theta = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta} \; , \tag{12}$$

so it isn't constant unless $\cos\theta = 0$ ($\theta = \pi/2$).

[Remaining bits are a good deal easier to see using the Hamiltonian approach, but any other valid method gets full marks.]

By writing the angular momentum in terms of the momenta,

$$J^2 = m^2 r^4 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \; , \tag{13}$$

we see that (using $\dot{p}_\phi = 0$)

$$2J\dot{J} = 2p_\theta \dot{p}_\theta - 2\dot{\theta} \frac{p_\phi^2 \cos\theta}{\sin^3\theta} \tag{14}$$

Now use $\dot{\theta} = \partial H / \partial p_\theta = p_\theta / mr^2$ and recall (12) above to see that $J$ is a constant.

When the potential has a dipole term $A\cos\theta / r^2$, we still have $\partial H / \partial \phi = 0$ so that $p_\phi$ is constant. We now get, however,

$$\dot{p}_\theta = \frac{p_\phi^2 \cos\theta}{\sin^3\theta} + \frac{A \sin\theta}{r^2} \Rightarrow \frac{\mathrm{d}J^2}{\mathrm{d}t} = \frac{2p_\theta A \sin\theta}{r^2} \; . \tag{15}$$

From the definition of $p_\theta$ we see that

$$\frac{\mathrm{d}J^2}{\mathrm{d}t} = 2mA \sin\theta \; \dot{\theta} = -\frac{\mathrm{d}}{\mathrm{d}t} \left( 2mA \cos\theta \right) \; , \tag{16}$$

so that $J^2 + 2mA\cos\theta$ is a new conserved quantity.

Q3. The form

$$S = \int \mathrm{d}t \; L = \int \mathrm{d}t \; \left( -\frac{m_0 c^2}{\gamma} - U(r) \right) \tag{17}$$

is the correct expression for the relativistic action because $\mathrm{d}t = \gamma \mathrm{d}\tau$, where $\tau$ is the invariant proper time. This form of the Lagrangian allows us to express the proper times of all the particles of the system in terms of the laboratory time $t$.

In polar coordinates we have

$$L = -m_0 c^2 \left( 1 - \frac{\dot{r}^2}{c^2} - \frac{r^2 \dot{\theta}^2}{c^2} \right)^{1/2} - U(r) \; . \tag{18}$$

The Lagrangian does not depend on $\theta$ so the (angular) momentum conjugate to $\theta$ is constant. This evaluates to $\gamma m_0 r^2 \dot{\theta} = J$. The Lagrangian does not depend on time explicitly, so the Hamiltonian is also a constant, equal to the total energy $m_0 c^2 \gamma + U(r) = E$.

We need an expression for something like $\mathrm{d}r/\mathrm{d}\theta$ which we can get from $\dot{r}/\dot{\theta}$. To manipulate these conservation laws, write them in the form

$$r^2 \dot{\theta}^2 = \frac{J^2}{m_0^2 r^2 \gamma^2} \; ; \quad \gamma^2 = \frac{(E - U(r))^2}{m_0^2 c^4} \tag{19}$$

Then use the definition of $\gamma$:

$$1 - \frac{\dot{r}^2}{c^2} - \frac{r^2 \dot{\theta}^2}{c^2} = \frac{1}{\gamma^2} \Rightarrow \frac{\dot{r}^2}{c^2} + \frac{r^2 \dot{\theta}^2}{c^2} = 1 - \frac{1}{\gamma^2} \; . \tag{20}$$

Now divide through by $\dot{\theta}^2 r^4 / c^2 = J^2 / (m_0^2 c^2 \gamma^2)$ to generate the required term $(\mathrm{d}r/\mathrm{d}\theta)^2$ on the LHS:

$$\frac{1}{r^4} \left( \frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 + \frac{1}{r^2} = \left( \frac{\mathrm{d}}{\mathrm{d}\theta} \left( \frac{1}{r} \right) \right)^2 + \frac{1}{r^2} = \frac{m_0^2 c^2 (\gamma^2 - 1)}{J^2} \; . \tag{21}$$

Finally substitute for $\gamma^2$ to get the required form

$$\left( \frac{\mathrm{d}}{\mathrm{d}\theta} \left( \frac{1}{r} \right) \right)^2 + \frac{1}{r^2} = \frac{(E - U(r))^2 - m_0^2 c^4}{J^2 c^2} \; . \tag{22}$$

Setting $u \equiv 1/r$ and $U = -Ku$ we see that the equation is of the form

$$\left( \frac{\mathrm{d}u}{\mathrm{d}\theta} \right)^2 + u^2 = \frac{K^2 u^2}{J^2 c^2} + \frac{2EKu + E^2 - m_0^2 c^4}{J^2 c^2} \; . \tag{23}$$

This can be easily be manipulated into the form given in the question by completing the square. Looking at the quadratic term in $u$, we see that $\alpha^2 = 1 - K^2/J^2 c^2$.

The orbit is a precessing ellipse provided that $J^2 c^2 > K^2$. Orbits that have angular momentum lower than this will certainly encounter the origin...

Q4. The inverse transform is

$$\rho(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}^3\vec{k}\ \tilde{\rho}(\vec{k}) \exp(-i\vec{k}\cdot\vec{r}) \tag{24}$$

The relation between the Fourier transforms is

$$|\vec{k}|^2 \tilde{\varphi} = \frac{\tilde{\rho}}{\epsilon_0} \tag{25}$$

so we can (in the absence of noise) find the potential via the relation

$$\varphi(\vec{r}) = \frac{1}{(2\pi)^3\epsilon_0} \int_{-\infty}^{\infty} \mathrm{d}^3\vec{k}\ \frac{\tilde{\rho}(\vec{k})}{|\vec{k}|^2} \exp(-i\vec{k}\cdot\vec{r}) \tag{26}$$

For the case $\rho(\vec{r}) = A\cos(Qx)$ for the layer $-t \leq z \leq t$, we have the Fourier transform

$$\tilde{\rho}(x,y,z) = \int_{-t}^{t} \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x\ A\cos(Qx) \exp(i(k_x x + k_y y + k_z z)) \tag{27}$$

Writing $\cos(Qx) = \frac{1}{2}(\exp(iQx) + \exp(-iQx))$, using $\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-ikx) = 2\pi\delta(k)$ for the $x$ and $y$ integrals and doing the $z$ integral explicitly, we find

$$\tilde{\rho}(\vec{k}) = (2\pi)^2 A\delta(k_y)\left(\delta(k_x - Q) + \delta(k_x + Q)\right) \frac{\sin(k_z t)}{k_z}\ . \tag{28}$$

The back-transform is only required for $y = z = 0$ (the potential is independent of $y$ anyway, but the variation in $z$ is quite interesting...), so, after the trivial $k_y$ integral, we have

$$\varphi(x,0,0) = \frac{A}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \mathrm{d}k_z \int_{-\infty}^{\infty} \mathrm{d}k_x \left(\delta(k_x - Q) + \delta(k_x + Q)\right) \frac{\sin(k_z t)}{k_z} \frac{\exp(-i(k_x x))}{k_x^2 + k_z^2}\ . \tag{29}$$

Doing the $k_x$ integral leaves

$$\varphi(x,0,0) = \frac{A\cos(Qx)}{\pi\epsilon_0} \int_{\infty}^{\infty} \mathrm{d}k_z\ \frac{\sin(k_z t)}{k_z(k_z^2 + Q^2)} = \frac{A\cos(Qx)}{\pi\epsilon_0} t^2 I(Qt)\ , \tag{30}$$

using the definition of $I(a)$ given.

To do the integral, you can either write $\sin k = (\exp(ik) - \exp(-ik))/2i$ and close over the top for the first term and underneath for the second one, or express it as $\Im(\exp(ik))$ and just use the pole at $k = ia$, which has residue $\exp(-a)/2a^2$. There is an slight subtlety with the pole at the origin, which has residue $1/a^2$, but only contributes $\pi i \times$ residue because it is exactly on the path of integration. Because there was so much pole-wiggling in the course we'll be lenient...

The final answer is $\epsilon_0\varphi(x,0,0) = A\cos(Qx)\left(1 - \exp(-Qt)\right)/Q^2$.

Q5. The propagator $G(x, x'; t)$ is used to express the wavefunction $\Psi(x, t)$ as an integral over the initial one $\Psi(x, 0)$:

$$\Psi(x,t) = \int \mathrm{d}x' \; G(x,x';t)\Psi(x',t) \; . \tag{31}$$

The propagator satisfies the equation

$$\hat{\mathcal{H}}G - \mathrm{i}\hbar\frac{\partial}{\partial t}G = \delta(x-x')\delta(t) \; , \tag{32}$$

so that the correct time evolution of $\Psi(x, t)$ is guaranteed.

If we have a complete set of eigenvctors $\hat{\mathcal{H}}\phi_n = E_n\phi_n$ and write

$$\Psi(x,t) = \sum_n \; c_n\phi_n e^{-\mathrm{i}E_n t/\hbar} \tag{33}$$

then the integral becomes

$$\Psi(x,t) = \int \mathrm{d}x' \sum_n \phi_n(x)\phi_n^*(x')e^{-\mathrm{i}E_n t/\hbar}\sum_m c_m\phi_m(x') = \sum_m c_m\phi_m(x)e^{-\mathrm{i}E_n t/\hbar} \tag{34}$$

The integrals $\int \mathrm{d}x'\phi_n^*\phi_m(x) = \delta_{mn}$, so the form of the propagator is verified.

Start from the Schrödinger equation for the propagator $G(x, x'); t)$ and make the analogy with a diffusion process with an imaginary "effective diffusion constant" $D = \mathrm{i}\hbar/m$. Discuss how, for an infinitesimal time interval, one can represent the solution as the product of two independent processes: one of the free "diffusion" and the other due to the modified potential. (Taking the case of a free quantum particle, with no potential, would be sufficient for this discussion.)

Divide the time-axis into small discrete intervals. Use the property of convolution, $G(a, b) = \int \mathrm{d}c \; G(a, c)G(c, b)$, and show how the propagator $G(a, b)$ can be represented as a sequence of integrals over $\mathrm{d}x_n$ (at each time $t_n$) of a product of diffusion-like propagators over each infinitesimal time step.

By formally going to an infinitely-fine discretisation, define a notation for the path integral and its measure $\mathcal{D}[x]$. Identify the continuous squared time-derivative in the exponent and arrive at the expression: $G(a, b) = \int \mathcal{D}[x] \exp(-\int \mathrm{d}t \; mv^2/2D) = \int \mathcal{D}[x] \; \exp(+(i/\hbar) \int \mathrm{d}t \; L)$.

Q6. Setting $\partial P/\partial t = 0$ gives

$$\frac{\partial P}{\partial q} + \alpha q P = \text{ constant} \tag{35}$$

The constant has to be zero, because the probability and its derivative must vanish at $q = \infty$. The integral is then

$$\frac{1}{P}\frac{\partial P}{\partial q} = -\alpha q \Rightarrow P \propto \exp(-\alpha q^2/2) \; , \tag{36}$$

which is a zero-mean Gaussian with variance $1/\alpha$.

It's slightly easier to take the logarithm of $P$ for the term on the LHS, so that

$$\frac{1}{P}\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\log\Delta - \frac{(q-Q)^2}{2\Delta}\right) = \frac{\dot{\Delta}}{2\Delta}\left(-1 + \frac{(q-Q)^2}{\Delta}\right) + \dot{Q}\frac{(q-Q)}{\Delta} \tag{37}$$

as required.

The other terms follow similarly:

$$\frac{1}{P}\frac{\partial^2 P}{\partial q^2} = \frac{1}{\Delta} + \frac{(q-Q)^2}{\Delta^2}\ ; \quad \frac{1}{P}\frac{\partial}{\partial q}(\alpha q P) = \alpha\left(1 - \frac{q(q-Q)}{\Delta}\right)\ . \tag{38}$$

Collecting up the terms we get

$$\frac{\dot{\Delta}}{2\Delta}\left(-1 + \frac{(q-Q)^2}{\Delta}\right) + \dot{Q}\frac{(q-Q)}{\Delta} = D\left(\frac{1}{\Delta} + \frac{(q-Q)^2}{\Delta^2} + \alpha - \alpha\frac{q(q-Q)}{\Delta}\right)\ . \tag{39}$$

Both sides have to be equal, so the coefficients of all powers of $q$ have to be the same. The term in $q^2$ gives

$$\frac{\dot{\Delta}}{2\Delta^2} = \frac{D}{\Delta}\left(\frac{1}{\Delta} - \alpha\right)\ , \tag{40}$$

which is (almost) one of the required equations for $\dot{\Delta}$. To get the rest easily, it's best to express the final term in (39) as $\alpha((q-Q)^2 + Q(q-Q))$ and collect powers of $(q-Q)$. The term linear in $(q-Q)$ is then seen to imply $\dot{Q} = -D\alpha Q$. Checking the other powers of $(q-Q)$, we confirm the previous form for $\dot{\Delta}$ and all terms cancel correctly.

The solution is $Q(t) = Q_0 \exp(-D\alpha t)$ and $\Delta(t) = (1 - \exp(-2\Delta\alpha t))/\alpha$, which is a really lovely example of how the Fokker-Planck equation can be used in non-equilibrium statistical mechanics.